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THE SMALLEST Ω-IRRATIONAL CW-COMPLEX

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1. Introduction

It has recently been shown [2] that there is a finite simply-connected CW-complex X whose loop space has an irrational Poincaré series, i.e.,

$$P_{\Omega X}(z) = \sum_{n=0}^{\infty} \mathrm{Rank}_Q\, H_n(\Omega X; Q) z^n$$

is not a rational function of z. A space with this property is said to be *Ω-irrational*. In this paper we show that any Ω-irrational CW-complex must have at least four cells, excluding the base point. We also construct explicitly a four-cell Ω-irrational space W with $\dim W = 6$ and prove that this dimension cannot be reduced without increasing the number of cells.

2. Lower bounds on the size of W

We call on several well-established facts in proving that a four-cell complex in dimension five or lower or a three-cell complex in any dimension is Ω-rational.

Theorem A. *Let X be a 1-connected CW-complex. There is a CW-complex Y with the rational homotopy type of X such that Y has one cell for each generator of $H_*(X; Q)$. If X is finite, Y has the same number or fewer cells in each dimension as X. Furthermore, Y may be constructed so that for each $n \geq 1$, the $(n+1)$-cells of Y are attached to the $(n-1)$-skeleton Y^{n-1}.*

This is a consequence of Sullivan's theory of minimal models [7].

Theorem B. *Let $f: SX \to SY$ be a map between two suspensions, with X and Y path-*

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connected, and let Z be the mapping cone of f. Z is Ω-rational if and only if the quotient Hopf algebra $H_*(\Omega SY; Q)/\langle \mathrm{im}(\Omega f)_*\rangle$ has rational Hilbert series.

Here $H_*(\Omega SY; Q)$ is viewed as a graded Hopf algebra over Q and $\langle \mathrm{im}(\Omega f)_*\rangle$ denotes the two-sided ideal generated by the image of $\bar{H}_*(\Omega SX; Q)$ under $(\Omega f)_*$. Theorem B is proved in [5].

Theorem C. *If X and Y are simply connected Ω-rational complexes, then $X \vee Y$ is Ω-rational.*

An exact formula for $P_{\Omega(X\vee Y)}(z)$ in terms of $P_{\Omega X}(z)$ and $P_{\Omega Y}(z)$ is given in [5].

Theorem D. *Let H be a free associative graded algebra, finitely generated over a field F, and let w be a single homogeneous element. The quotient algebra $G = H/\langle w\rangle$ of H by the two-sided ideal w generates has a rational Hilbert series.*

This is proved in [3].

An immediate corollary of these is:

Lemma 1. *Let $f: S^m \to \bigvee_{i=1}^k S^{d_i}$ with $m \geq 2$, $d_i \geq 2$, and let Z be the mapping cone of f. Then Z is Ω-rational.*

Proof. If $\omega \in H_{m-1}(\Omega S^m; Q)$ generates the ring $H_*(\Omega S^m; Q)$, Theorem B implies that Z is Ω-rational if

$$H_*\left(\Omega \bigvee_{i=1}^{k} S^{d_i}; Q\right)/\langle(\Omega f)_*(\omega)\rangle$$

has a rational Hilbert series. This follows from Theorem D, since by [4] $H_*(\Omega \bigvee_{i=1}^k S^{d_i}; Q)$ is a free associative algebra.

Lemma 2. *A 1-connected CW-complex X with three or fewer (positive dimensional) cells is Ω-rational.*

Proof. Since

$$\pi_k(S^n)\otimes Q = \begin{cases} Q & \text{if } k=n, \text{ or } k=2n-1 \text{ and } n \text{ is even,} \\ 0 & \text{otherwise,} \end{cases}$$

Theorem A implies that the only rational homotopy types for a two-cell complex Y are (pt), $S^{m_1} \vee S^{m_2}$, and $X_0 = S^{2n} \cup_g e^{4n}$, where the attaching map for the last has non-zero Hopf invariant. Each of these is Ω-rational by Lemma 1. If there is a third cell, say in dimension $m = \dim X$, it must be attached to one of these complexes. The only case not covered by Lemma 1 is that of a mapping cone of $f: S^{m-1} \to X_0$. The only non-trivial rational homotopy of X_0 in dimensions $4n-1$ and above is

$\pi_{6n-1}(X_0)\otimes Q \approx Q$, generated, say, by $[h]$. Letting $\simeq$ denote rational homotopy equivalence, we have either $X \simeq X_0 \vee S^m$, whence Ω-rationality follows from Theorem C, or $X \cong X_0 \cup_{\lambda \cdot h} e^{6n}$ for some scalar λ. In the latter case, the Serre spectral sequence in cohomology for the path fibration over X gives

$$P_{\Omega X}(z) = (1 + z^{2n-1})(1 - z^{8n-2})^{-1}.$$

Lemma 3. *Let X be a finite* 1*-connected* CW*-complex with* $\dim X \le 5$. *Then X has the rational homotopy type of a mapping cone of a map between two wedges of spheres,*

$$f: \bigvee_{j=1}^{r} S^{m_j} \to \bigvee_{i=1}^{k} S^{d_i},$$

where each $m_j = 3$ *or* 4 *and each* $d_i = 2$ *or* 3.

Proof. This follows directly from Theorem A. We may assume that the 4- and 5-cells of X are attached directly to the 3-skeleton, which in turn is obtained by attaching cells to the base point.

Lemma 4. *Let X be a* 1*-connected* CW*-complex with* $\dim X \le 5$. *Suppose that* $\bar{H}_*(X; Q)$ *has four or fewer generators, e.g., if X has four of fewer cells. Then X is Ω-rational.*

Proof. Combining Theorem A and Lemma 3, we see that X is a mapping cone as in Lemma 3, where $r + k \le 4$. By Theorem B, the rationality of $P_{\Omega X}(z)$ is equivalent to the rationality of the Hilbert series of a certain associative algebra G, which has a finite presentation with k generators and r relations. The generators occur in degrees one or two and the relations in degrees two or three. If $r = 1$, apply Lemma 1. If $k = 1$, G is a (commutative) quotient of $Q[x]$ and has a rational Hilbert series. The remaining possibility is $r = k = 2$, which we now explore.

Let the generators of G be x and y and call the relations α and β. We may assume that α is not a multiple of β and vice versa in the free algebra $H = Q\langle x, y\rangle$, since otherwise G would have a smaller presentation. By a slight refinement of the proof of Lemma 3, taking into account the fact that each S^{m_j} maps to the $(m_j - 1)$-skeleton of $\bigvee_{i=1}^{k} S^{d_i}$, we can assume that α and β are decomposable. This means that α and β belong to H_+^2, where H_+ consists of the positive degree elements of H. Since they are images of generators of homology of spheres, α and β are primitive in the free Hopf algebra H. The severe restrictions on what α and β may be make it feasible to consider all cases.

If $\deg x = \deg y = 2$, there are no three-dimensional elements for α and β to be. If $\deg x = 1$, $\deg y = 2$, the only possibility is $\alpha = x^2$, $\beta = [x, y]$, whence $G = Q[x, y]/(x^2)$ has series $(1 - z)^{-1}$. If $\deg x = \deg y = 1$, a primitive relation in degree two or three must be a linear combination of either $\{x^2, [x, y], y^2\}$ or $\{[x^2, y], [x, y^2]\}$. Let $K = Q\langle x, y\rangle / \langle [x^2, y], [x, y^2]\rangle$. K is the universal enveloping algebra of L, $K = U(L)$, where L is the positively graded Lie algebra spanned by $\{x, y, x^2, y^2, [x, y]\}$ with higher brackets vanishing. As α and β cannot be multiples, $\langle\alpha, \beta\rangle$ contains $\langle [x^2, y], [x, y^2]\rangle$, and $G = Q\langle x, y\rangle/\langle\alpha, \beta\rangle$ is either K or a quotient of K. Either way, G is the universal enveloping algebra of a nilpotent graded Lie algebra, implying that G has a rational Hilbert series.

In view of Lemmas 2 and 4, the smallest Ω-irrational complex must have at least four cells. If it has precisely four cells, it must have dimension six or greater.

3. Construction of the minimal complex

The remainder of the paper is devoted to demonstrating the existence of a four-cell complex W with $P_{\Omega W}(z)$ irrational and $\dim W = 6$. Ufnarovsky [8] has given an example of a Hopf algebra with irrational Hilbert series. His example has two generators, in degrees two and four, and two relations, in degrees ten and fourteen. It can be used to construct a four-cell Ω-irrational complex with cells in dimensions 3, 5, 12, and 16. The minimal complex W presented here was inspired by Ufnarovsky's and has certain similarities to it.

W is the mapping cone of a map $f = f_1 \vee f_2 : S^5 \vee S^5 \to S^2 \vee S^2$, which we now specify. Let x and y denote the generators of $\pi_2(S^2 \vee S^2, x_0) \approx \mathbb{Z} \oplus \mathbb{Z}$ as well as the corresponding generators of $H_1(\Omega(S^2 \vee S^2); Q)$. The attaching maps f_1 and f_2 are given by the Whithead products

$$[f_1] = [x, [x, [x, y]]] \quad \text{and} \quad [f_2] = [[[x, y], y], y].$$

Let α and β denote the corresponding elements of $H_4(\Omega(S^2 \vee S^2); Q)$. In view of Theorem B, it suffices to show that the graded Hopf algebra $G = Q\langle x, y\rangle/\langle\alpha, \beta\rangle$ has an irrational Hilbert series.

We will show that $G = U(L)$, where L is a certain positively graded Lie algebra over Q having rank r_n in dimension n, where $r_n = 2 - \cos(\pi n/2)$ for $n \geq 1$. The Hilbert series of G is then an infinite product

$$\begin{aligned} P(z) &= \left[\frac{(1+z)^{r_1}}{(1-z^2)^{r_2}}\right]\left[\frac{(1+z^3)^{r_3}}{(1-z^4)^{r_4}}\right]\left[\frac{(1+z^5)^{r_5}}{(1-z^6)^{r_6}}\right]\cdots \\ &= \left[\frac{(1+z)^2}{(1-z^2)^3}\right]\left[\frac{(1+z^3)^2}{(1-z^4)}\right]\left[\frac{(1+z^5)^2}{(1-z^6)^3}\right]\cdots \\ &= \left[\left(\frac{1}{1-z}\right)^2\right]\left[\left(\frac{1}{1-z^2}\right)\right]\left[\left(\frac{1}{1-z^3}\right)^2\right]\left[\left(\frac{1}{1-z^4}\right)\right]\cdots. \end{aligned}$$

$P(z)$ is easily seen to be a transcendental function, since it converges in the open unit disk, yet

$$\lim_{z \to 1} (z-1)^k P(z) = \infty$$

for any finite k. We may also compute $P(z)$ by the Jacobi triple product identity ([1], Theorem 2.8) to obtain the interesting but clearly irrational series

$$P(z)^{-1} = 1 - 2z + 2z^4 - 2z^9 + \cdots + (-1)^n(2)z^{n^2} + \cdots.$$

We shall describe L, and only at the end observe that $U(L)$ has the presentation which defines G. To define L, we will make use of the correspondence between positively graded Lie algebras and commutative differential graded algebras [6]. We describe a specific differential graded algebra $(\Lambda V, d)$, where V is a graded vector space with rank r_n in dimension $n+1$, and compute $H^k(\Lambda V, d)$ for $k=1$ and $k=2$. Here k refers to the homological or 'wedge' grading of ΛV, so each Q-module $H^k(\Lambda V, d)$ is still graded by dimension. Since $(H^k(\Lambda V, d))_m$ agrees with $\mathrm{Ext}_{U(L)}^{k, m-k}(Q, Q)$ for the corresponding L, we will have found the minimal presentation we seek for $U(L)$.

A basis for V in dimension $2k$ $(k \geq 1)$ we denote as $\{a_{2k-1}, b_{2k-1}\}$. In dimension $4k-1$, V is spanned by $\{p_{4k-2}, q_{4k-2}, c_{4k-2}\}$ and in dimension $4k+1$ by $\{c_{4k}\}$. The choice of subscripts corresponds to the degrees of the duals in L, and it implies that d, which has bidegree $(+1, +1)$, actually preserves subscript sums on any homogeneous element of ΛV.

Since the differential d satisfies $d(xy) = d(x)y + (-1)^{\deg x} x d(y)$, we need only specify d on a basis for V. For $k \geq 1$, set

$$d(a_1) = d(b_1) = 0,$$

$$d(p_2) = a_1^2, \qquad d(q_2) = b_1^2, \qquad d(c_2) = a_1 b_1,$$

$$d(a_{4k-1}) = -\sum_{i=1}^{2k-1} a_{2i-1} c_{4k-2i} + \sum_{i=1}^{k} p_{4i-2} b_{4k-4i+1},$$

$$d(b_{4k-1}) = \sum_{i=1}^{2k-1} (-1)^i b_{2i-1} c_{4k-2i} + \sum_{i=1}^{k} q_{4i-2} a_{4k-4i+1},$$

$$d(c_{4k}) = \sum_{i=1}^{k} p_{4i-2} q_{4k-4i+2} + \sum_{i=1}^{2k} (-1)^i a_{2i-1} b_{4k-2i+1},$$

$$d(a_{4k+1}) = \sum_{i=1}^{2k} (-1)^i a_{2i-1} c_{4k-2i+2} - \sum_{i=1}^{k} p_{4i-2} b_{4k-4i+3},$$

$$d(b_{4k+1}) = \sum_{i=1}^{2k} b_{2i-1} c_{4k-2i+2} - \sum_{i=1}^{k} q_{4i-2} a_{4k-4i+3},$$

$$d(p_{4k+2}) = a_{2k+1}^2 + 2\sum_{i=0}^{k-1} a_{2i+1} a_{4k-2i+1} + 2\sum_{i=0}^{k-1} p_{4i+2} c_{4k-4i},$$

$$d(q_{4k+2}) = b_{2k+1}^2 + 2\sum_{i=0}^{k-1} b_{2i+1} b_{4k-2i+1} - 2\sum_{i=0}^{k-1} q_{4i+2} c_{4k-4i},$$

$$d(c_{4k+2}) = \sum_{i=0}^{2k} a_{2i+1} b_{4k-2i+1}.$$

It is a straightforward but tedious calculation, which we omit, to verify that $d^2=0$. Supplementing the total degree and homological degree gradings, we make ΛV into a trigraded algebra by setting

$$e(a_{2k-1})=1, \qquad e(p_{4k-2})=2, \qquad e(b_{2k-1})=-1, \qquad e(q_{4k-2})=-2,$$

$e(c_{4k-2})=e(c_{4k})=0$. The above formulas show that d preserves e-degree, so each $H^k(\Lambda V,d)$ will be bigraded, by total degree and e-degree.

$H^1(\Lambda V,d)$ has rank two, with $\{a_1,b_1\}$ being a basis. This is clear since all generators occur in distinct bidegrees and d vanishes only on a_1 and b_1.

4. Computation of $H^2(\Lambda V,d)$

Computing $H^2(\Lambda V,d)$ is somewhat more complicated. We shall prove that $H^2(\Lambda V,d)$ is zero except in bidegrees $(6,\pm 2)$, where it has rank one. The generators of $H^2(\Lambda V,d)$ are $p_2c_2+a_3a_1$ and $q_2c_2+b_3b_1$. Let

$$d'=d\,|_V: V\to\Lambda^2V \quad\text{and}\quad d''=d\,|_{\Lambda^2V}:\Lambda^2V\to\Lambda^3V.$$

Because d respects the trigrading of ΛV, we may compute $H^2(\Lambda V,d)$ by considering $\ker d''$ and $\operatorname{im} d'$ restricted to each bidegree separately.

In bidegree $(m,n)\neq(6,\pm 2)$, $\operatorname{im} d'$ has rank either zero or one and we wish to show that $\ker d''$ has the same rank. In $(6,\pm 2)$, $\operatorname{im} d'=0$ and we will see that rank $(\ker d'')=1$. Use the bases consisting of monomials for Λ^2V and Λ^3V, and assume that all vectors are expressed as linear combinations of monomials. $\gamma_x(t)$ denotes the coefficient of the basis element x in the linear combination t. If A is a basis for a vector space M, we say that $x\in A$ *occurs in* t if and only if $\gamma_x(t)\neq 0$.

To show rank$(\ker d'')=0$ in a particular bidegree we will let $\tau\in\ker d''$ and show that no monomial of Λ^2V occurs in τ. When we want to show that rank$(\ker d'')\leq 1$, we will show that some monomials do not occur in an arbitrary $\tau\in\ker d''$ and that those which do occur must have their coefficients in some fixed ratios to one another. This implies that any non-zero element of $\ker d''$ is a scalar multiple of any other, i.e., rank$(\ker d'')\leq 1$.

We make repeated use of the following three lemmas, which also introduce some shorthand notation. In them, $\phi:M\to N$ is a linear transformation between two vector spaces over a field F. A denotes a basis for M and B is a basis for N.

Lemma 5. *If $y\in B$ occurs in $\phi(x_1)$ for some $x_1\in A$ but does not occur in any $\phi(x)$ for $x\in A-\{x_1\}$, then $\gamma_{x_1}(\tau)=0$ for any $\tau\in\ker\phi$. We describe this situation by the shorthand "$y\Rightarrow x_1:0$".*

Lemma 6. *Suppose $x_1,x_2\in A$, $y\in B$, and y occurs in $\phi(x_1)$ and $\phi(x_2)$ but not in $\phi(x)$ for $x\in A-\{x_1,x_2\}$. Let $\lambda=-\gamma_y(\phi(x_2))\gamma_y(\phi(x_1))^{-1}$. Then $\gamma_{x_1}(\tau)=\lambda\gamma_{x_2}(\tau)$ for any $\tau\in\ker\phi$. This we express as "$y\Rightarrow x_1:\lambda x_2$". If in addition we know that $\gamma_{x_1}(\tau)=0$*

for any $\tau \in \ker \phi$, *we may deduce that* $\gamma_{x_2}(\tau)=0$ *for all* $\tau \in \ker \phi$. *This situation is denoted* "$(y; x_1) \Rightarrow x_2 : 0$".

Lemma 7. *Suppose* $x_1, x_2, x_3 \in A$, $y \in B$, *and* y *occurs in* $\phi(x_i)$, $i=1,2,3$, *but not in* $\phi(x)$ *for* $x \in A-\{x_1, x_2, x_3\}$. *Let* $\lambda = -\gamma_y(\phi(x_2))\gamma_y(\phi(x_1))^{-1}$. *If* $\gamma_{x_3}(\tau)=0$ *for all* $\tau \in \ker \phi$, *then* $\gamma_{x_1}(\tau)=\lambda\gamma_{x_2}(\tau)$ *for any* $\tau \in \ker \phi$. *The shorthand for this is* "$(y; x_3) \Rightarrow x_1 : \lambda x_2$". *The converse also holds and we express it by* "$(y; x_1, x_2) \Rightarrow x_3 : 0$".

Proofs. Lemmas 5, 6, and 7 all follow from the observation that whenever $\tau \in \ker \phi$,

$$0=\gamma_y(0)=\gamma_y(\phi(\tau))=\gamma_y\left(\sum_{x\in A}\gamma_x(\tau)\phi(x)\right)$$

$$=\sum_{x\in A}\gamma_x(\tau)\gamma_y(\phi(x))=\sum_{\substack{x\in A\\ \gamma_y(\phi(x))\neq 0}}\gamma_x(\tau)\gamma_y(\phi(x)).$$

The notation "$x_1 : \lambda x_2$" by itself means that $\gamma_{x_1}(\tau)=\lambda\gamma_{x_2}(\tau)$ whenever $\tau \in \ker \phi$. Note that $x_1 : \lambda x_2$ and $x_2 : \lambda' x_3$ together imply $x_1 : \lambda\lambda' x_3$. Now we take $\phi = d''$ and see what happens in each possible bidegree. k is any positive integer. Lemmas 5, 6, and 7 are relevant because each cubic monomial of $\Lambda^3 V$ occurs in at most three of the d-images of quadratic monomials.

In bidegree $(4k+2, 4)$, $\operatorname{im} d'=0$.

$$a_1 a_{4i-3} p_{4k-4i+2} \Rightarrow p_{4i-2} p_{4k-4i+2} : 0, \quad 1 \le i \le k/2.$$

In bidegree $(5,3)$, $\operatorname{im} d'=0$ and $a_1^3 \Rightarrow p_2 a_1 : 0$.

In bidegree $(4k+1, 3)$, $k \ge 2$, $\operatorname{im} d'=0$.

$$a_3 a_{4i-5} a_{4k-4i+1} \Rightarrow p_{4i-2} a_{4k-4i+1} : 0 \quad \text{if } 2 \le i \le k.$$

$$(a_1^2 a_{4k-3};\, p_{4k-2} a_1) \Rightarrow p_2 a_{4k-3} : 0.$$

In bidegree $(4k+3, 3)$, $\operatorname{im} d'=0$.

$$a_1 a_{4i-3} a_{4k-4i+3} \Rightarrow p_{4i-2} a_{4k-4i+3} : 0, \quad 1 \le i \le k.$$

In bidegree $(6,2)$, $\operatorname{im} d'=0$. $a_1^2 c_2 \Rightarrow a_3 a_1 : p_2 c_2$. A basis for $(\Lambda^2 V)_{(6,2)}$ is $\{a_3 a_1, p_2 c_2\}$. Since $d''(a_3 a_1 + p_2 c_2)=0$, $\ker d'' = \operatorname{span}(a_3 a_1 + p_2 c_2)$, and $(\ker d''/\operatorname{im} d')_{(6,2)}$ has rank one.

In bidegree $(4k+2, 2)$, $k \ge 2$, $\operatorname{im} d'=0$.

$$p_2 c_{4i-4} c_{4k-4i+2} \Rightarrow p_{4i-2} c_{4k-4i+2} : 0, \quad 2 \le i \le k.$$

$$(p_{4k-2} a_1 b_1;\, p_{4k-2} c_2) \Rightarrow a_1 a_{4k-1} : 0.$$

$$(a_1^2 c_{4k-2};\, a_1 a_{4k-1}) \Rightarrow p_2 c_{4k-2} : 0.$$

$$(a_{2i-1}^2 c_{4k-4i+2};\, p_{4i-2} c_{4k-4i+2}) \Rightarrow a_{2i-1} a_{4k-2i+1} : 0, \quad 1 < i \le k.$$

In bidegree (4, 2), $\operatorname{im} d' = \operatorname{span}(a_1^2) = (\Lambda^2 V)_{(4,2)} = \ker d''$.
In bidegree (4k, 2), $k \geq 2$, $\operatorname{rank}(\operatorname{im} d') = 1$.

$$p_2 p_{4i-2} q_{4k-4i-2} \Rightarrow p_2 c_{4k-4} : p_{4i-2} c_{4k-4i}, \quad 1 < i < k.$$

$$p_2 a_{4i-1} b_{4k-4i-3} \Rightarrow p_2 c_{4k-4} : a_{4i-1} a_{4k-4i-1}, \quad 1 \leq i < k/2.$$

$$p_2 a_{4i-3} b_{4k-4i-1} \Rightarrow p_2 c_{4k-4} : a_{4i-3} a_{4k-4i+1}, \quad 1 \leq i \leq k/2.$$

$$p_2 a_{2k-1} b_{2k-3} \Rightarrow p_2 c_{4k-4} : 2a_{2k-1}^2.$$

In bidegree (5, 1), $\operatorname{rank}(\operatorname{im} d') = 1$ and $a_1^2 b_1 \Rightarrow a_1 c_2 : -p_2 b_1$.
In bidegree (4k + 1, 1), $k \geq 2$, $\operatorname{rank}(\operatorname{im} d') = 1$.

$$a_1 a_{4i-1} b_{4k-4i-1} \Rightarrow a_1 c_{4k-2} : a_{4i-1} c_{4k-4i}, \quad 1 \leq i < k.$$

In particular, $a_1 c_{4k-2} : a_3 c_{4k-4}$.

$$a_1 c_2 c_{4k-4} \Rightarrow a_3 c_{4k-4} : a_{4k-3} c_2.$$

$$a_3 a_{4i+1} b_{4k-4i-5} \Rightarrow a_3 c_{4k-4} : a_{4i+1} c_{4k-4i-2}, \quad 1 \leq i < k-1.$$

So $a_1 c_{4k-2} : a_{2i-1} c_{4k-2i}$ for any i, $1 \leq i < 2k$. Lastly,

$$a_{2i-1}^2 b_{4k-4i+1} \Rightarrow a_{2i-1} c_{4k-2i} : -p_{4i-2} b_{4k-4i+1}, \quad 1 \leq i \leq k.$$

In bidegree (4k + 3, 1), $\operatorname{rank}(\operatorname{im} d') = 1$.

$$a_1 p_{4i-2} q_{4k-4i+2} \Rightarrow a_1 c_{4k} : p_{4i-2} b_{4k-4i+3}, \quad 1 \leq i \leq k.$$

$$a_1 a_{4i-1} b_{4k-4i+1} \Rightarrow a_1 c_{4k} : -a_{4i-1} c_{4k-4i+2}, \quad 1 \leq i \leq k.$$

In particular, $a_1 c_{4k} : -a_3 c_{4k-2}$.

$$a_3 a_{4i+1} b_{4k-4i-3} \Rightarrow a_3 c_{4k-2} : -a_{4i+1} c_{4k-4i}, \quad 1 \leq i < k.$$

In bidegree (4, 0), $\operatorname{im} d' = \operatorname{span}(a_1 b_1) = (\Lambda^2 V)_{(4,0)} = \ker d''$.
In bidegree (4k, 0), $k \geq 2$, $\operatorname{rank}(\operatorname{im} d') = 1$.

$$a_1 a_{4i-1} a_{4k-4i-2} \Rightarrow a_1 b_{4k-3} : a_{4i-1} b_{4k-4i-1}, \quad 1 \leq i < k.$$

In particular, $a_1 b_{4k-3} : a_3 b_{4k-5}$.

$$a_3 a_{4i-3} q_{4k-4i-2} \Rightarrow a_3 b_{4k-5} : a_{4i-3} b_{4k-4i+1}, \quad 1 \leq i < k.$$

$$p_2 b_1 b_{4k-5} \Rightarrow a_3 b_{4k-5} : a_{4k-3} b_1.$$

$$(a_1 c_{4i-4} b_{4k-4i+1}; a_1 b_{4k-3}, a_{4i-3} b_{4k-4i+1}) \Rightarrow c_{4i-4} c_{4k-4i+2} : 0, \quad 2 \leq i \leq k.$$

In bidegree (6, 0), $\operatorname{rank}(\operatorname{im} d') = 1$.

$$a_1^2 q_2 \Rightarrow p_2 q_2 : -a_1 b_3 \quad \text{and} \quad p_2 b_1^2 \Rightarrow p_2 q_2 : a_3 b_1.$$

In bidegree (4k + 2, 0) for $k \geq 2$, we need a more sophisticated argument.

$$a_{2i-1}c_{2k}b_{2k-2i+1} \Rightarrow a_{2i-1}b_{4k-2i+1} : (-1)^k a_{2k+2i-1}b_{2k-2i+1}, \quad 1 \le i \le k.$$

$$a_{2i-1}^2 q_{4k-4i+2} \Rightarrow a_{2i-1}b_{4k-2i+1} : (-1)^i p_{4i-2}q_{4k-4i+2}, \quad 1 \le i \le k.$$

By considering the three monomials in whose d''-images $x = a_1 b_{2i-1} c_{4k-2i}$ occurs, we get

$$\gamma_x d''(a_1 b_{4k-1}) = (-1)^i, \qquad \gamma_x d''(a_{4k-2i+1}b_{2i-1}) = 1,$$

$$\gamma_x d''(c_{2i}c_{4k-2i}) = (-1)^{i+1} \quad \text{if } i \ne k.$$

From this it follows that, for $1 \le i < k$,

$$-\gamma_{a_1 b_{4k-1}}(\tau) + (-1)^i \gamma_{a_{4k-2i+1}b_{2i-1}}(\tau) + \gamma_{c_{2i}c_{4k-2i}}(\tau) = 0$$

for any $\tau \in \ker d''$, hence

$$\gamma_{p_2 q_{4k-2}}(\tau) - \gamma_{p_{4k-4i+2}q_{4i-2}}(\tau) + \gamma_{c_{2i}c_{4k-2i}}(\tau) = 0,$$

when $\tau \in \ker d''$, $1 \le i < k$. A similar consideration of the monomials in whose d''-images $p_2 q_{4i-2} c_{4k-4i}$ occurs gives, for $1 \le i < k$ and $i \ne k/2$,

$$2\gamma_{p_2 q_{4k-2}}(\tau) - 2\gamma_{p_{4k-4i+2}q_{4i-2}}(\tau) + \gamma_{c_{4i}c_{4k-4i}}(\tau) = 0, \quad \tau \in \ker d''.$$

These two equations together give $2\gamma_{c_{2i}c_{4k-2i}}(\tau) = \gamma_{c_{4i}c_{4k-4i}}(\tau)$ for $\tau \in \ker d''$, i.e., $c_{4i}c_{4k-4i} : 2c_{2i}c_{4k-2i}$ for $1 \le i < k$, $i \ne k/2$. If k is even and $i = k/2$, we are considering coefficients of $c_k c_{3k}$. But $p_2 q_{2k-2} c_{2k} \Rightarrow p_2 q_{4k-2} : p_{2k+2} q_{2k-2}$, whence

$$a_1 b_{4k-1} : (-1)^{k/2} a_{3k+1} b_{k-1},$$

and

$$(a_1 b_{k-1} c_{3k};\, a_1 b_{4k-1}, a_{3k+1} b_{k-1}) \Rightarrow c_k c_{3k} : 0.$$

By considering the subscripts of 'c' to be modulo $4k$, we see that $c_k c_{3k} : 0$ when k is even and that $c_{4i}c_{4k-4i} : 2c_{2i}c_{4k-2i}$ for any i, except $i \equiv 0,\ \pm k/2,\ k \pmod{2k}$. Furthermore,

$$c_{2^s i}c_{-2^s i} : 2c_{2^{s-1}i}c_{-2^{s-1}i} : \cdots : 2^{s-1}c_{2i}c_{4k-2i}$$

for $s \ge 1$ as long as $2^{s-1}i \not\equiv 0$ or $k \pmod{2k}$. Now suppose $1 \le i < k$ with $i \ne k/2$. If $2k \mid 2^s i$ for some $s \ge 1$, let s be the smallest integer with this property. We must have $s \ge 3$ and $2^{s-1}i \equiv k \pmod{2k}$, so k is even and $c_k c_{3k} : \pm 2^{s-2} c_{2i}c_{4k-2i}$, implying (since char $Q \ne 2$) $c_{2i}c_{4k-2i} : 0$. If instead $2^s i \not\equiv 0 \pmod{2k}$ for any s, choose $s \ge 1$ and $r > s$ such that $2^r i \equiv 2^s i \pmod{2k}$. Then

$$c_{2^r i}c_{-2^r i} : 2^{r-s} c_{2^s i}c_{-2^s i} = 2^{r-s} c_{2^r i}c_{-2^r i}.$$

I.e., the coefficient of $c_{2^r i}c_{-2^r i}$ in any $\tau \in \ker d''$ equals 2^{r-s} times itself, and this can only happen in Q if that coefficient is zero. $c_{2^r i}c_{-2^r i} : 0$, and since $c_{2^r i}c_{-2^r i} : 2^{r-1}c_{2i}c_{4k-2i}$, we have $c_{2i}c_{4k-2i} : 0$ as well. We have shown that $c_{2i}c_{4k-2i} : 0$ for any i, $1 \le i < k$.

Using this information we easily obtain fixed ratios among the coefficients of the other monomials of $(\Lambda^2 V)_{(4k+2,0)}$.

$$(a_1 b_{4k-2i-1} c_{2i}; c_{2i} c_{4k-2i}) \Rightarrow a_1 b_{4k-1} : (-1)^i a_{2i+1} b_{4k-2i-1}, \quad 1 \le i < k;$$

we already have $a_{2i-1} b_{4k-2i+1} : (-1)^k a_{2k+2i-1} b_{2k-2i+1}$, so

$$a_1 b_{4k-1} : (-1)^i a_{2i+1} b_{4k-2i-1} \quad \text{for } 1 \le i < 2k.$$

$a_1^2 q_{4k-2} \Rightarrow a_1 b_{4k-1} : -p_2 q_{4k-2}$ and

$$(p_2 q_{4k-4i-2} c_{4i}; c_{4k-4i} c_{4i}) \Rightarrow p_2 q_{4k-2} : p_{4i+2} q_{4k-4i-2}, \quad 1 \le i < k, \ i \ne k/2.$$

We have seen $p_2 q_{4k-2} : p_{2k+2} q_{2k-2}$, for k even.

To deal with the elements of negative e-degree, we make use of a certain symmetry in ΛV. Define an involution $\mu : \Lambda V \to \Lambda V$ by

$$\mu(a_{2k-1}) = b_{2k-1}, \qquad \mu(b_{2k-1}) = a_{2k-1},$$

$$\mu(p_{4k-2}) = q_{4k-2}, \qquad \mu(q_{4k-2}) = p_{4k-2}, \qquad \mu(c_{2k}) = (-1)^{k+1} c_{2k},$$

and extend μ to a homomorphism of rings. It is easy to check that $d\mu = \mu d$, so μ induces an isomorphism of cohomology which negates e-degree. $H^*(\Lambda V, d)$ is symmetric in e-degree around zero, so $H^2(\Lambda V, d)$ must have just two generators, in bidegrees $(6, \pm 2)$.

This completes the proof that a bigraded Lie algebra L exists with two generators in bidegrees $(1, \pm 1)$ and two relations in bidegrees $(4, \pm 2)$. We need only confirm that $U(L)$ has the presentation stated earlier. Calling the generators x and y (with $e(x) = +1$, $e(y) = -1$), the relations must be $[x, [x, [x, y]]]$ and $[[[x, y], y], y]$, since these and their scalar multiples are the only elements of the proper bidegrees in the free Lie algebra on $\{x, y\}$.

Note added in proof

It has since come to my attention that the facts we deduced about the Lie algebra L are also implicit in the paper by Kac and Vinberg, Adv. in Math. 30 (1978) 137–155.

References

[1] G.E. Andrews, Theory of Partitions (Addison Wesley, Reading, MA, 1976).
[2] D. Anick, A counterexample to a conjecture of Serre, Ann. of Math. 115 (1982) 1-33.
[3] J. Backelin, La Série de Poincaré-Betti d'une algèbre graduée de type fini à une relation est rationnelle, C.R. Acad. Sci. Paris Serie A287 (1978) 843-846.
[4] I.M. James, Reduced product spaces, Ann. of Math. 62 (1955) 170–197.
[5] J.-M. Lemaire, Algèbres Connexes et Homologie des Espaces de Lacets, Lecture Notes in Mathematics 422 (Springer, Berlin-New York, 1974).
[6] D. Quillen, Rational homotopy theory, Ann. of Math. 90 (1969) 205-295.
[7] D. Sullivan, Infinitesimal computations in topology, Inst. Hautes Etudes Sci. Publ. Math. 47 (1978) 269-331.
[8] V.A. Ufnarovsky, On algebras' growth, Vestnik Moscow. Univ. Ser. I Mat. Meh. (1978) (4) 59-65.